Free Vibration of Thin Cylindrical Shells

V I WEINGARTEN*
Northiop Corporation, Hawthorne, Calif

The free vibrations of finite cylindrical shells are investigated With the aid of a number of simplifying assumptions, a frequency equation based on the known chaiacteristic functions for beams with any combination of boundary conditions is obtained Experimental results for frequency spectra and mode shapes of a cylinder fixed on one edge and free on the other are in good agreement with both Rayleigh's inextensional theoiy and the approximate fiequency equation Structural damping coefficients obtained for the test cylinders are compared with those of previous investigations

Nomenclature

a	=	radius of cylinder
b	=	parameter $[12\rho a^4(1 - \nu^2)/Eh^2]$
c^2	=	geometry parameter $[h^2/12(1 - \nu^2)a^2]$
D	=	flexural stiffness of cylinder wall $[Eh^3/12(1 - \nu^2)]$
E	=	Young s modulus of cylinder material
f	=	frequency, cps ($f = \omega/2\pi$)
$f_n'g$	=	viscous damping coefficient [see Eq (A1)]
h	=	thickness of the cylinder
l	=	length of the cylinder
$m - 1$	=	number of nodes in axial mode shape
n	=	number of circumferential waves
p	=	internal or external pressure
t	=	time
w	=	radial deflection
x	=	distance along the longitudinal axis of the cylinder
θ	=	angle denoting the circumferential location of a point on the cylinder middle surface
ξ	=	nondimensional axial coordinate (x/a)
λ_{kmn}	=	characteristic roots [see Eq (3)]
ρ	=	density of the shell material
ω	=	circular frequency ($2\pi f$)
Ω	=	frequency parameter $[12\rho a^4(1 - \nu^2)\omega^2/Eh^2]$
∇^2	=	$[(\partial^2/\partial x^2) + (1/a^2)](\partial^2/\partial\theta^2)$
$\bar{\nabla}^2$	=	$(\partial/\partial\xi^2) + (\partial/\partial\theta^2)$

I Introduction

THE free vibration of a cylindrical shell has interested many investigators In 1894, Lord Rayleigh[1] derived an approximate expression for the natural frequencies of vibration of a cylindrical shell based on a separation of the effects of bending and stretching A later treatment by Love[2] resulted in a general dynamical theory of shells which included both bending and extensional deformations. Love's equations were first used by Flugge[3] to obtain a cubic frequency equation for a simply supported cylinder, a result which indicated that there were three frequencies for each nodal pattern A more detailed investigation made by Arnold and Warburton[4 5] showed that the three frequencies corresponded to essentially radial, axial, and circumferential vibrations with the radial vibration frequency much lower than the other two Their analysis also showed that the natural frequency may decrease as the number of circumferential waves increases, in contrast with the results of inextensional theory Arnold and Warburton also investigated the natural frequencies of cylinders clamped at both edges, with the use of the Rayleigh-Ritz method

Recent investigators have concentrated on simplifying the method of analysis of vibrating cylindrical shells By means of a number of approximations, Yu[6] was able to obtain a simple expression for the radial frequencies of a clamped or simply supported cylinder vibrating in a mode consisting of a number of circumferential waves that is large compared to the number of axial waves Simplified frequency equations were also obtained by Vlasov,[7] Breslavskii,[8] and Reissner[9] by neglecting the circumferential and axial inertia forces of the shell Finally, the simplifications of Breslavskii and Yu were combined by Rapoport[10] to yield frequency equations for a shell with various boundary conditions

In the present paper, a method similar to Rapoport's has been used An experimental investigation of the frequency spectra and mode shapes of a clamped-free cylinder was also performed The experimental data are in good agreement with theory Structural damping was investigated as a secondary part of the experimental program Viscous damping coefficients were obtained for each resonance point of the supported-free cylinder and tabulated as a function of wave shape and frequency The results are compared with those of previous investigations in the Appendix

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* Member of the Technical Management, Norair Division

II Approximate Method of Analysis

The well-known Donnell differential equation of a circular cylindrical shell under an external radial loading p can be written as[11]

$$D\nabla^8 w + \frac{Eh}{a^2}\frac{\partial^4 w}{\partial x^4} - \nabla^4 p = 0 \tag{1}$$

This equation can be applied to vibration problems of cylindrical shells when we assume that the circumferential and longitudinal inertia forces are negligible Then the exteinal loading p can be replaced by the radial inertia force

$$-\rho h(\partial^2 w/\partial t^2)$$

Upon substituting this value into Eq (1) and nondimensionalizing the resulting equation, we obtain

$$\bar{\nabla} w^8 + \frac{1}{c^2}\frac{\partial^4 w}{\partial \xi^4} + b\frac{\partial^2}{\partial t^2}\bar{\nabla}^4 w = 0 \tag{2}$$

Let us assume w to be of the form

$$w = (\sum_k c_k e^{i\lambda_{kmn}\xi})\cos n\,\theta\,\sin\omega t \tag{3}$$

Equation (3) will satisfy Eq (2) if the coefficients λ_{kmn} are the roots of the following equation:

$$\Omega = \frac{(\lambda_{kmn}^2 + n^2)^4 + [(1/c^2)\lambda_{kmn}^4]}{(\lambda_{kmn}^2 + n^2)^2} \tag{4}$$

which is an eighth-order equation for λ_{kmn} as a function of c, n, and Ω Conversely, we note that, if any one of these

Table 1 Frequency comparsion of experimental results with the "approximate" and Arnold and Warburton theories[a]

$m \backslash n$		3	4	5	6	7	8	9	10	11	12	13	14
1	Exp	1025	700	(545) (559)	525	(587) (598)	720	885	(1090) (1100)	1310	1560	1850	2140
	Approx	1431	872	629	565	617	739	905	1101	1323	1569	1837	2127
	A-W	1220	796	594	541	595	718	883	1079	1301	1547	1815	2105
2	Exp		1620	1210	980	(838) (875)	900	995	(1135) (1145)	1365	(1555) (1600)	1865	2160
	Approx		2084	1460	1118	964	949	1034	1186	1384	1616	1877	2162
	A-W		2086	1453	1106	952	938	1024	1178	1376	1609	1870	2156
3	Exp				1650	1395	1350	(1260) (1295)	1325	(1460) (1470)	(1680) (1700)	(1900) (1930)	2210
	Approx		3434	2503	1911	1551	1366	1319	1380	1520	1717	1955	2228
	A-W		3130	2332	1812	1490	1322	1284	1348	1489	1687	1925	2197
4	Exp					1960	1865		1690	1730	1830	2020	2260
	Approx				2800	2268	1928	1746	1695	1751	1888	2088	2335
	A-W				2845	2294	1942	1752	1697	1750	1888	2087	2334

Frequency measured in cycles per second Cylinder properties: material = steel radius = 3 in $l/a = 4$ $a/h = 300$

roots is known, then the frequency parameter Ω is determined

Since the exact determination of a root λ_{mn} is quite difficult, a number of simplifying assumptions will be made to obtain approximate values Let us assume that n^2 is large compared to λ_{mn}^2 Then Eq (4) can be approximated by

$$\lambda_{kmn}^4 = c^2(\Omega n^4 - n^8) \quad (5)$$

For given values of n, c, and Ω the right-hand side of Eq (5) is a constant Therefore, four approximate values of λ_{kmn} are of the form

$$\lambda_{mn}, -\lambda_{mn}, i\lambda_{mn}, -i\lambda_{mn} \quad (6)$$

The remaining four values of λ_{kmn} implied by Eq (3) are neglected

The deflection function w can now be written approximately as

$$w = (c_1 \sin\lambda_{mn}\xi + c_2 \cos\lambda_{mn}\xi + c_3 \sinh\lambda_{mn}\xi + c_4 \cosh\lambda_{mn}\xi) \cos n\theta \sin\omega t \quad (7)$$

which gives a longitudinal deflection shape similar to that of the vibrating beam Approximate values of λ_{mn} are now obtained by substituting Eq (7) into the appropriate boundary condition equations for a vibrating beam and solving the resulting determinant The characteristic roots λ_{mn} obtained by the foregoing procedure are identical to the vibrating beam characteristic roots A tabulation of these values, for various combinations of boundary conditions, can be found in Ref 12 and in many textbooks (for example, Refs 13 and 14) The frequency parameter Ω for a given n is now

Table 2 Tabulation of experimental values of frequency spectra, node locations, and viscous damping coefficients

Mat—1020 Steel $h = 0\ 010$ in $a/h = 400$ $a/k = 0\ 448$

m	n	Inextensional theory f, cps	Experimental data f, cps	Approximate theory Cl -free f, cps	Approximate theory S S free f, cps	$f'g$	S_1	S_2	S_3	S_4	S_5
1	3	46	400	422	54	1 06	0				
	5	140	239	219	151	0 73	0				
	7	280	304	310	296	0 75	0				
	8	368	376	396	387	0 79	0				
	10	577	595	610	605	1 40	0				
	11	700	713	737	733	1 19	0				
	12	834	844	876	872	2 03	0				
	13	978	992	1027	1023		0				
2	7		837	742	590		0	0 75			
	8		693	666	561	1 22	0	0 68			
	9		642	665	596	1 10	0	0 7			
	12		855	931	908		0	0 40			
	13		992	1071	1053	2 00	0	0 46			
	14		1161	1228	1213	2 13	0	0 51			
	15		1375	1399	1386	1 98	0	0 71			
	16		1373	1584	1573	1 65	0	0 56			
	17		1746	1782	1772	2 26	0	0 70			
3	6		1905	2070	1850	3 00	0	0 50	0 86		
	11		1084	1098	1021	1 46	0	0 44	0 85		
	13		1219	1227	1182	2 17	0	0 35	0 76		
	14		1344	1349	1313	1 77	0	0 39	0 85		
	16		1640	1668	1642	1 83	0	0 48	0 87		
	17		1816	1855	1833	2 57	0	0 45	0 84		
4	6		3250	3245	3105	3 05	0	0 35	0 66	0 80	
	7		3083	2689	2547	2 95	0	0 25	0 64	0 91	
	8		2208	2266	2131	1 97	0	0 38	0 63	0 88	
	12		1598	1541	1460	1 70	0	0 27	0 66	0 94	
	14		1513	1600	1543	1 63	0	0 25	0 65	0 82	
	15		1679	1700	1652	1 83	0	0 32	0 56	0 84	
5	6		4060	4246	4207	2 80	0	0 23	0 45	0 60	0 80
	14		1985	1974	1910	1 88	0	0 10	0 40	0 66	0 89
	17		2184	2220	2174	1 80	0	0 15	0 40	0 62	0 89

obtained by substituting the real values of the beam characteristic roots λ_{mn} into Eq (4) and solving directly for Ω

Although the method outlined is based on heuristic reasoning, its justification is that the results obtained from it are in good agreement with experimental results As an initial check, results of the approximate frequency equation for a cylinder with clamped ends are compared in Table 1 with experimental frequencies obtained by Koval[15] and with the results of another approximate equation obtained by Arnold and Warburton[5] Arnold and Warburton used an energy approach to obtain an approximate solution for the frequencies of a cylindrical shell clamped at both ends Even though Arnold and Warburton only used a one-term approximation for the axial mode shape, their analytical results agreed very closely with the experimental results given in Ref 5 In general, the results of the present approximate theory are greater than those obtained by the Arnold and Warburton equation but are in about as good agreement with experimental results A comparison of these results is shown in Table 1 From Table 1 we see that the agreement between the present approximate theory and experiment becomes better as the number of circumferential waves increases, in accordance with the assumptions of the theory

III Experimental Investigation

A series of tests was performed on cylinders with one end clamped and the other free, as an additional check of the approximate method outlined in Sec II Since a detailed discussion of the test setup and test procedure can be found in Ref 16, only a brief outline will be given here

The two test specimens used were made of 1020 steel with dimensions as given in Table 1 The cylinders were formed over an 8-in -diam mandrel, and the seam was formed by a butt weld The cylinders were then spun to eliminate eccentricities One end of the cylinders was clamped in an aluminum plate which contained a trough filled with cerrobend, a low melting point alloy The other end of the cylinder was free The cylinder was supported by a shaft attached to the end plate as shown in Fig 1

An electromagnet was used to excite the specimen The test procedure consisted of varying the frequency of the electromagnet by means of an oscillator until a resonant frequency was reached The frequency was accurately measured by an electronic counter A microphone, which could traverse the cylinder axially and circumferentially, measured the response of the shell The microphone output and the

Table 2 (Continued)

		Mat—1020 Steel		$h = 0\ 040$ in		$a/h = 100$		$a/l = 0\ 448$			
				Approximate theory							
m	n	Inextensional theory f, cps	Experimental data f cps	Cl free f cps	S S -free f, cps	$f'g$	S_1	S_2	S_3	S_4	S_5
1	2	68	320	910	96	1 42	0				
	3	186	332	486	218	0 900	0				
	4	350	402	470	387	0 860	0				
	5	560	559	639	605	0 485	0				
	6	815	795	892	872	0 365	0				
	7	1120	1081	1203	1187	0 345	0				
	8	1473	1414	1565	1551	0 387	0				
	9	1870	1794	1976	1963	0 38	0				
	10	2310	2211	2436	2423	0 82	0				
2	5		1186	1432	1167	0 965	0	0 710			
	6		1156	1335	1168	0 88	0	0 700			
	7		1311	1468	1364	0 300	0	0 710			
	8		1582	2118	1675	0 550	0	0 710			
	11		2678	3057	3017	1 12	0	0 220			
	13		3762	4212	4157		0	0 140			
	14		4350	4865	4829		0	0 130			
3	2		5782	6106	6056		0	0 325	0 540		
	3		4225	4719	4504	3 55	0	0 310	0 760		
	5		2547	2850	2594	3 25	0	0 390	0 810		
	6		2162	2401	2178	1 22	0	0 420	0 800		
	7		2007	2227	2047	1 24	0	0 380	0 780		
	8		2081	2284	2143	1 00	0	0 430	0 840		
	9		2316	2518	2406		0	0 380	0 700		
	10		2630	2877	2785	1 01	0	0 420	0 820		
	12		3742	3853	3781	1 12	0	0 230	0 720		
	14		5014	5082	5017						
	15		5316	5777	5714		0	0 380	0 810		
	16		6001	6523	6462	3 01	0	0 375	0 790		
4	2		6883	6937	7104	3 73	0	0 225	0 525	0 890	
	3		5713	5941	5962	3 36	0	0 230	0 440	0 790	
	4		4766	4977	4897	3 30	0	0 175	0 550	0 780	
	6		3377	3593	3433	3 25	0				
	7		3033	3240	3077	1 37	0	0 22	0 51	0 83	
	8		2871	3112	2957	1 48	0	0 27	0 56	0 85	
	9		2949	3184	3044	1 00	0	0 29	0 56	0 85	
	10		3159	3423	3297	1 38	0	0 25	0 54	0 86	
	11		3495	3792	3678	1 36	0	0 28	0 57	0 87	
	12		3936	4261	4156	2 01	0	0 27	0 56	0 85	
	13		4429	4811	4712	1 71	0	0 30	0 56	0 85	
	16		6310	6845	6756		0	0 28	0 53	0 87	
5	8		4069	4042	3916	1 47	0	0 24	0 47	0 65	0 88
	10		4117	4144	4009	1 41	0	0 20	0 43	0 68	0 90
	11		3743	4426	4294		0	0 20	0 45	0 65	0 88
	16		6728	7288	7172	2 70	0	0 23	0 40	0 56	0 88

geometrical position of the microphone were recorded on an x-y plotter to yield a graphical plot of the longitudinal and circumferential mode shape for a given resonance frequency A typical record is shown in Fig 2

IV Comparison of Clamped-Free Cylinder Test Results with Theory

The numerical and experimental results of the present investigation are given in Table 2 For the first longitudinal mode of vibration ($m = 1$), the results of the approximate theory and of Rayleigh's inextensional theory are compared with the experimental results in Figs 3 and 4 Numerical results of the approximate theory for both a simply supported-free cylinder and a clamped-free cylinder are plotted All of the theoretical curves are very close to each other for the larger values of n It is interesting to note that the experimental values follow the approximate curve for the clamped-free cylinder very closely for all values of n for the cylinder with a radius thickness ratio of 400 The experimental results for the cylinder with a radius-thickness ratio of 100, on the other hand, fall between the theoretical curves for the clamped-free cylinder and the simply supported-free cylinder for low values of n ($n \leq 4$) The reason for this discrepancy is suspected to be imperfect clamping and is being investigated further by means of additional tests and by a more accurate theory

A comparison, in Figs 5 and 6, of the results of the approximate equation with experimental results for $m \geq 2$ indicates good agreement The results also indicate that the value of n at which the minimum frequency occurs depends upon the axial wavelength As m increases, the value of n corresponding to the minimum frequency also increases This phenomenon was first noted by Arnold and Warburton for cylinders that were simply supported or clamped at both ends

The position of the experimentally determined axial node locations are designated as S_i in Table 2 The position of

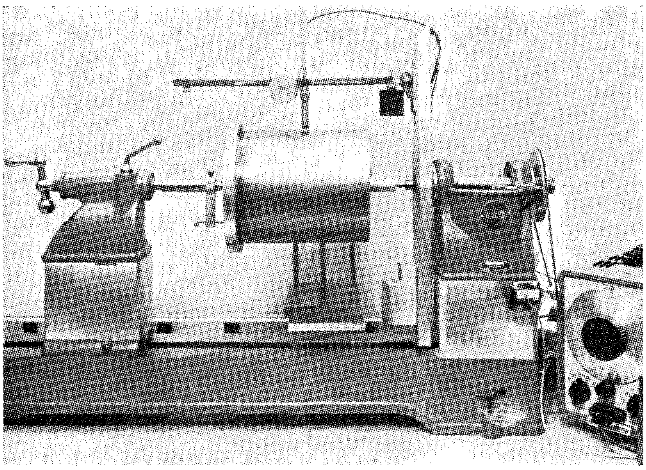


Fig 1 Over-all view of test setup

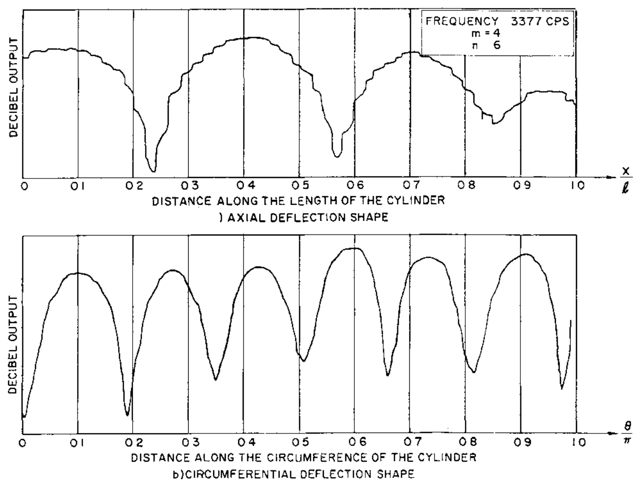


Fig 2 A typical mode shape ıecord

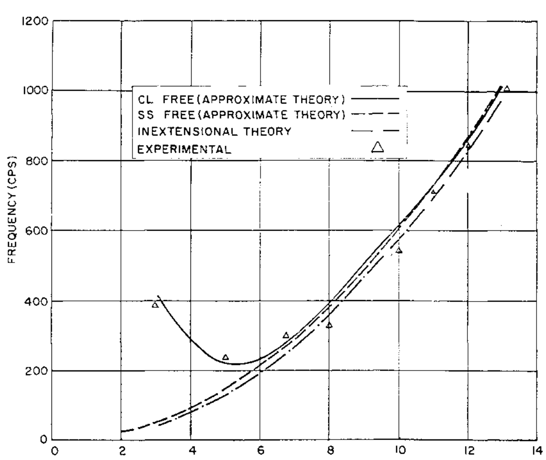


Fig 3 Graphical comparison of experimental results with theory for $m = 1$ of a clamped-free shell ($a/h = 400$)

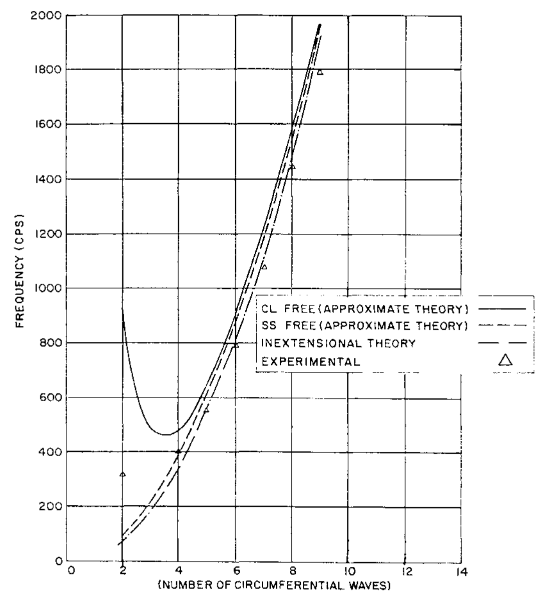


Fig 4 Graphical comparison of experimental results with theory for $m = 1$ of a clamped-free shell ($a/h = 100$)

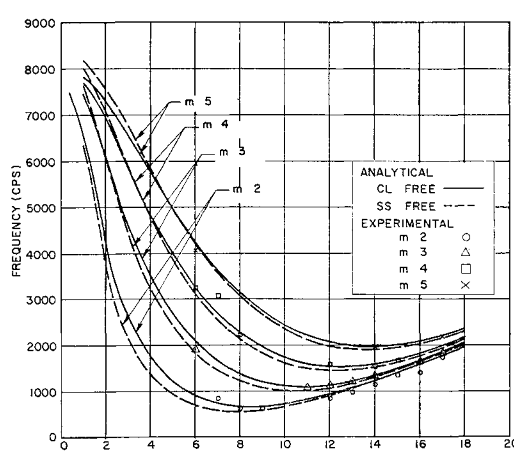


Fig 5 Graphical comparison of experimental results with theory for $m > 1$ of a clamped-free shell ($a/h = 400$)

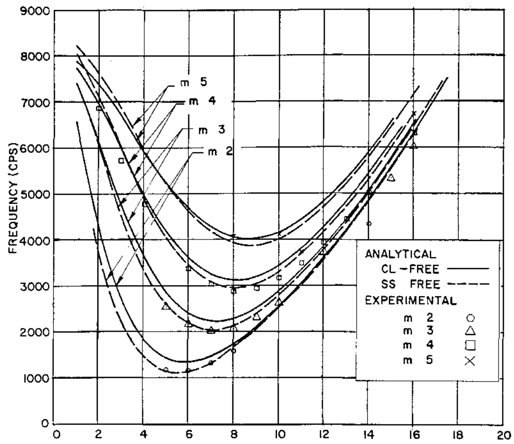


Fig 6 Graphical comparison of experimental results with theory for $m > 1$ of a clamped-free shell ($a/h = 100$)

clamped-free and simply supported-free beam node points are tabulated in Table 3 The variation of the position of the experimental node points for different values of n is greatest at $m = 2$ and decreases as the number of axial waves increases The tabulated results of the average experimental axial node positions correspond fairly well with the beam node positions but do not coincide exactly, a result probably due to the fact that the beam functions are not exact solutions of the equations for the vibrations of cylindrical shells

V Conclusions

Experimental and theoretical results for clamped-free and clamped-clamped cylinders are in good agreement for larger values of n (say, $n > 4$) It appears, therefore, that the approximate frequency equation can be used for arbitrary boundary conditions in this region For $n > 4$ and $m = 1$, the Rayleigh inextensional theory gives reasonable results for the clamped-free cylinders The experiments also indicate that, although the beam functions may not be the true deflection shape, they are near enough to the true shape so that when used in conjunction with the frequency equation they give a very close approximation to the experimental data Some anomalies in the agreement between theory and experiment were obtained for modes characterized by $n < 4$ These are being investigated further at the present time

Appendix: Structural Damping

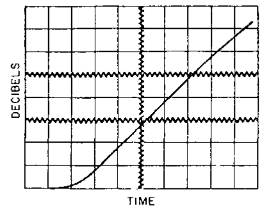
In addition to the frequency spectra, over-all structural damping coefficients were experimentally obtained These coefficients were obtained by passing the response signal coming from the microphone through an a c -d c log converter and recording the output on an oscilloscope camera The output was calibrated by a decibel meter and the decay curves obtained by disconnecting the electromagnet from the circuit while the cylinder was at resonance A typical decay curve appears in Fig 7

Table 3 Position of clamped free and simply supported free beam node points[a]

m	S_1	S_2	S_3	S_4	S_5
1	0				
2	0	0 73/0 79			
3	0	0 45/0 51	0 85/0 87		
4	0	0 31/0 35	0 61/0 65	0 89/0 91	
5	0	0 23/0 27	0 47/0 49	0 71/0 73	0 93/0 93

[a] Key = S S free/Cl free

Fig 7 Typical decay cuıve



The equation of motion of the free vibration of an elastic system (using the same notation as given in Ref 17) can be written as

$$\ddot{\phi}_n + c_n\dot{\phi}_n + \omega_n{}^2\phi_n = 0(c_n = 2\pi f_n{}'g) \qquad \text{(A1)}$$

where

g = structural damping coefficient
$f_n{}'g$ = viscous damping coefficient
ϕ_n = amplitude of nth mode
ω_n = natural frequency corresponding to ϕ_n

Tabulated values of $f'g$ as a function of mode shape and frequency are given in Table 2 A plot of the "viscous" damping coefficient as a function of the number of circumferential waves is shown in Fig 8 The results indicate a large scatter with $f_n{}'g = 2$ being an average value for both cylinders It is interesting to note that an average value of two for the viscous damping coefficient is close to that found by the author in Ref 16 for a steel cylinder clamped on both ends Fung, Sechler, and Kaplan obtained an average value $f_n{}'g = 6$ for a set of aluminum cylinders Figure 8 also shows that, for $n < 8$, the $m = 1$ and $m = 2$ modes have an average value of one for $f'g$ and the $m = 4$ and $m = 5$ modes an average value of three The $m = 3$ modes seem to fluctuate between one and three The peaking effect at a unique frequency found by the author in Ref 16 did not occur in the present investigation It is clear that much work remains to be done before the structural damping phenomenon is completely understood

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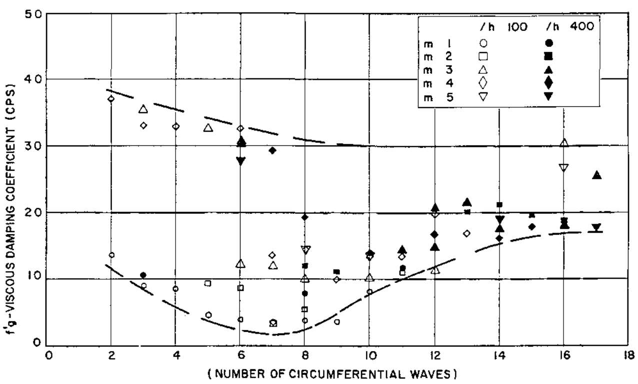

Fig 8 Viscous damping coefficient as a function of number of circumferential waves

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Determination of Dominant Error Sources in an Inertial Navigation System by Iterative Weighted Least Squares

W EISNER* AND A F GOODMAN†
North American Aviation, Inc , Anaheim, Calif

A critical problem in inertial navigation system evaluation is the determination of the dominant system error sources by an analysis of flight test data For both cruise and ballistic systems, this problem is reduced to estimating the coefficients in a linear model from data corrupted by nonstationary noise A procedure is presented which estimates the coefficients when the noise is nonstationary and/or correlated The procedure estimates the coefficients by least squares; then it iteratively obtains estimates of the necessary noise variances and covariances, and uses this information to re-estimate the coefficients by weighted least squares Two IBM 7090 or 7094 FORTRAN computer programs have been written to implement the procedure for nonstationary, uncorrelated noise An illustrative example is included

Introduction

THE use of inertial guidance equipment may conveniently be divided into two major areas, namely, cruise applications and ballistic applications Cruise systems are characterized by operating times that are significant compared to the Schuler 84-min period and, generally, are subjected to relatively low accelerations Typical applications in navigation are for aircraft, ships, submarines, and cruise (air-breathing) missiles Ballistic systems are characterized by shorter flight times compared to the Schuler period (i e , 5 min or less) and relatively high accelerations The inertial system error sources, which act as the dominant factors in determining system performance, differ in the two applications In cruise systems, gyro errors such as random drift and mass unbalance dominate In ballistic systems, uncertainty of accelerometer scale factor and accelerometer nonlinearities tend to be the dominant errors A critical problem in inertial navigation system evaluation is, then, the determination of the dominant system error sources

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* Manager, Systems Engineering Section, Systems Division, Autonetics
† Senior Technical Specialist, Navigation Systems Division, Autonetics

Ultimately, there is no substitute for flight testing of inertial systems Laboratory tests and such techniques as centrifuge testing and sled testing can supply much useful information; however, effects of sustained linear accelerations, aircraft maneuvers, complex vibration, and the acoustic environment acting upon the entire system cannot be adequately duplicated on the ground

The following paragraphs briefly discuss the error models for cruise and ballistic systems and present a procedure to determine the dominant system error sources by an analysis of flight test data

Cruise System Error Model

It has been shown[1,2] that the propagation of the position error of a cruise autonavigator is given by

$$(d^2\overline{\Delta R}/dt^2) + 2\bar{\Omega} \times (d\overline{\Delta R}/dt) + \omega^2\overline{\Delta R} = \bar{D} \qquad (1)$$

where $\overline{\Delta R}$ is the vehicle position error vector, ω is the Schuler frequency, $[\omega = (g/a)^{1/2}]$ $\bar{\Omega}$ is the earth angular velocity vector, $\bar{D}$ is the error driving function, and a is the radius of the earth $\bar{D}$ may be generated by such error sources as initial platform misalignment, gyro drift, initial position error, and accelerometer bias shift and scale factor errors For shorter operating times compared to the 24-hr period